The Card Game Zero sumZ and its Mathematical Cousins

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Zero sumZ was created with…

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- **Mathieu Guay-Paquet, University of Waterloo**

Alejandro

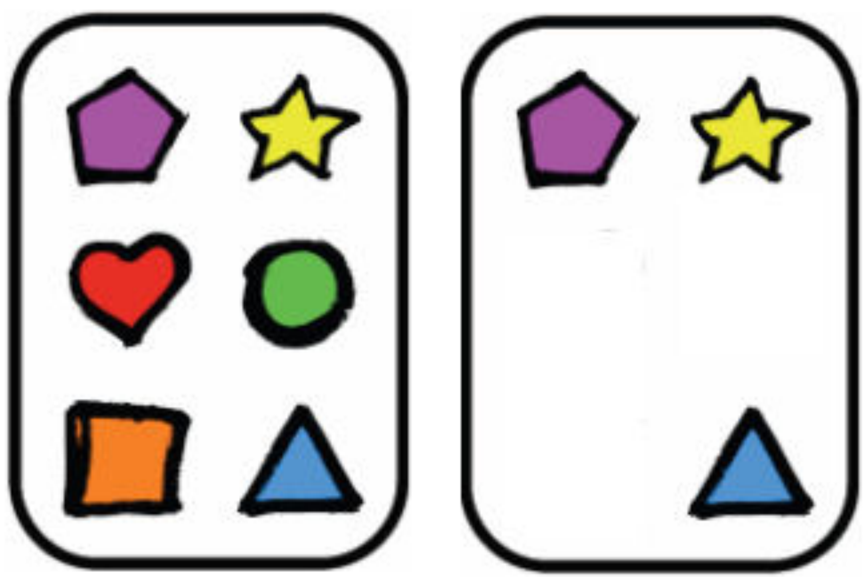
Mathieu

Outline

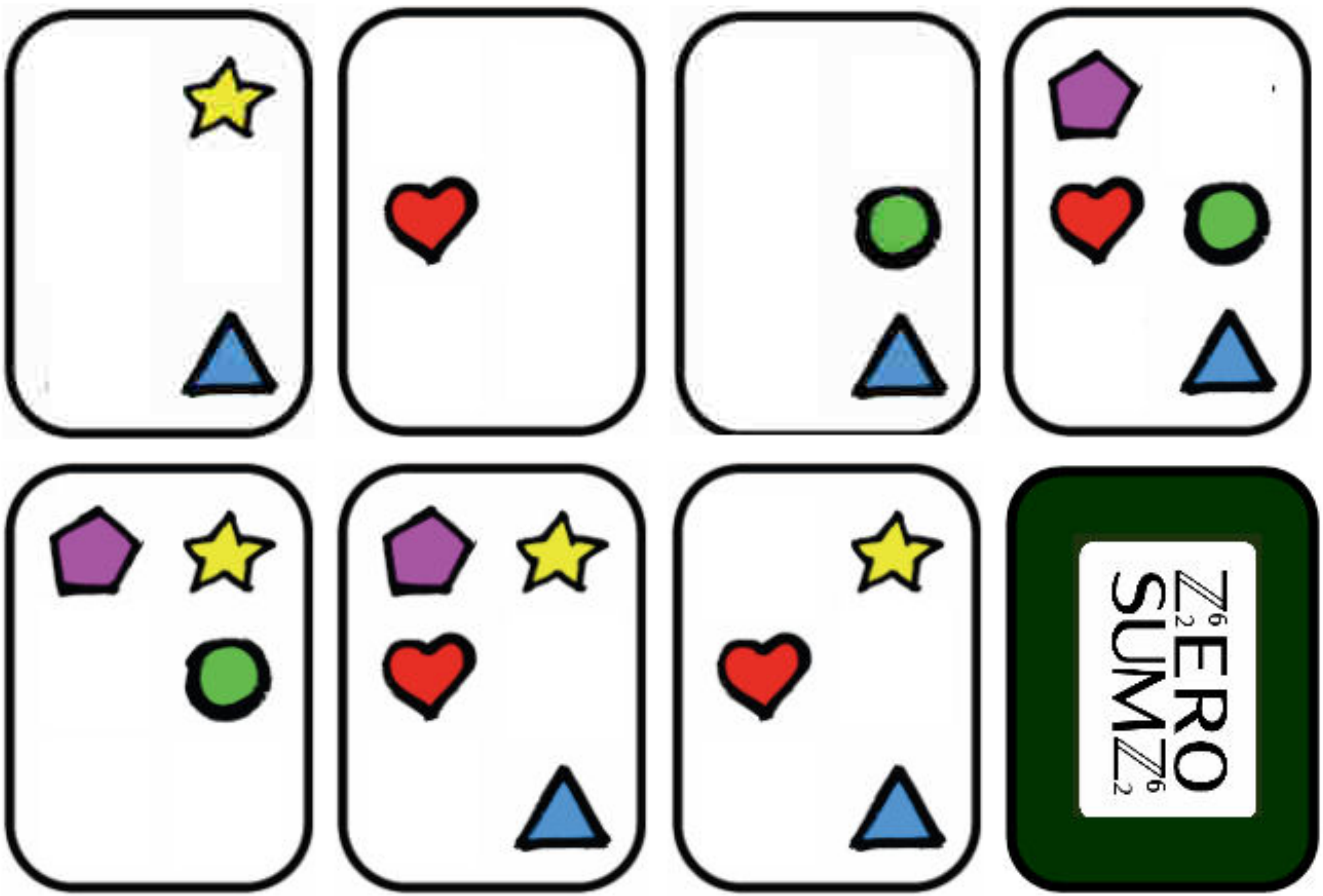
- Rules of the Game
- A Couple of Practice Hands
- Connection to the Card Game Set
- Interesting Facts (Theorems?) about Set
- Interesting Facts about Zero sumZ / Open Problems
- $$ Game $$
- A “Grandmaster” Strategy in Zero sumZ
- Other Game Variations

Rules of the Game

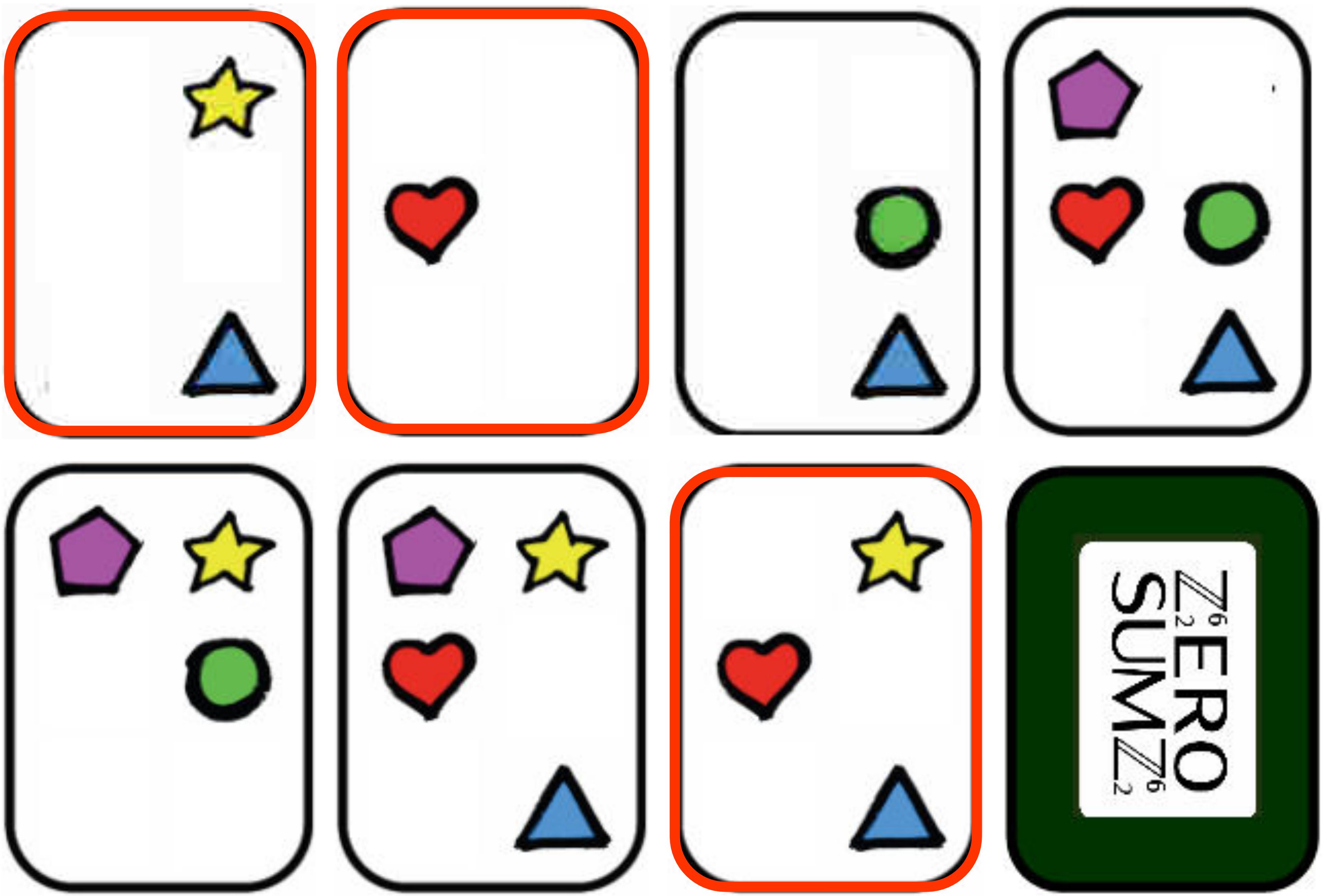
- The game is played by two or more players (though also possible to play solitaire).
- Seven cards are dealt and the first person to find a “zero set” gets to keep the cards comprising the set.
- New cards are dealt to fill in for the cards just removed, and another round is played.
- Play continues until all cards are dispensed. The player with the most cards at the end wins.



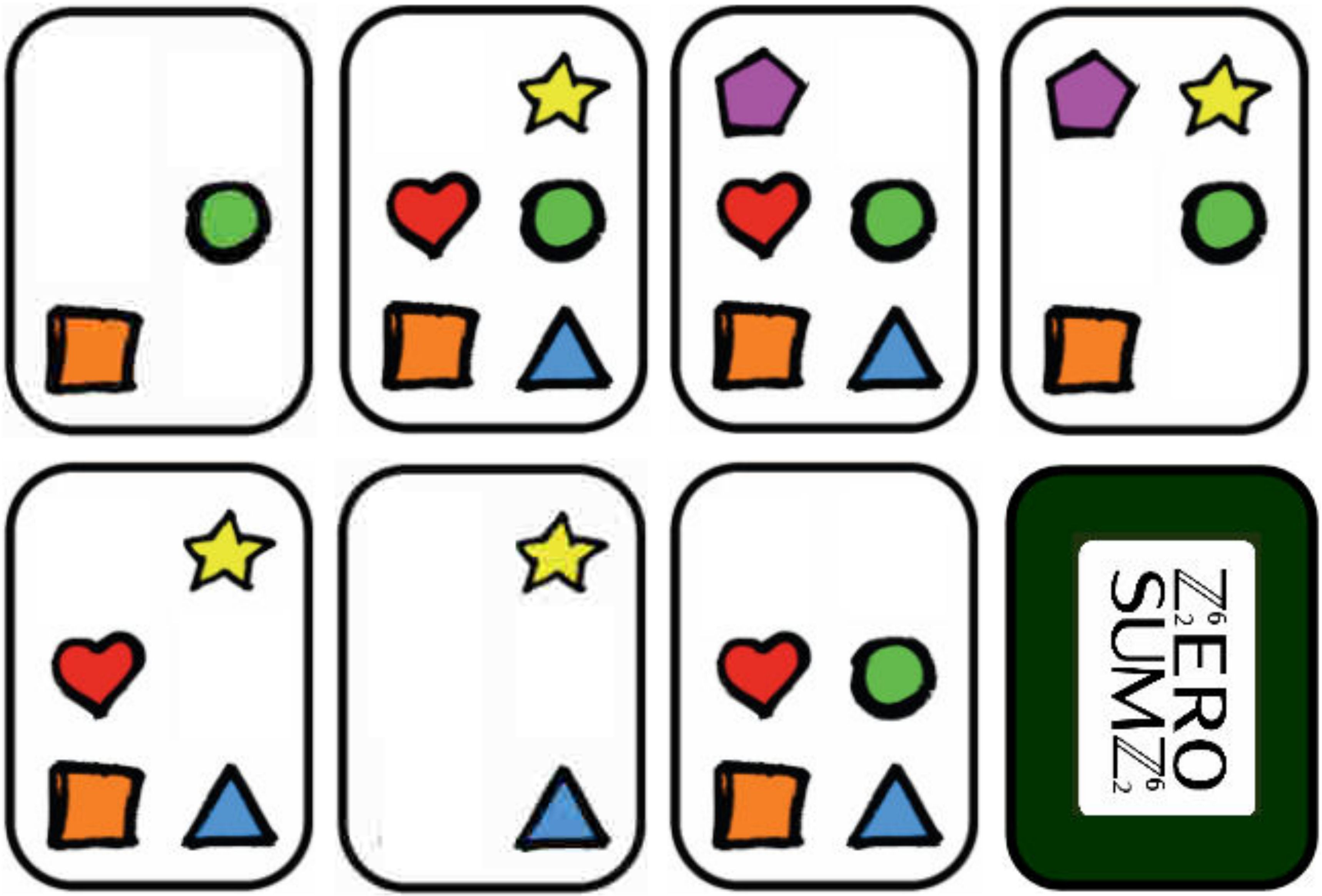
Learning the Rules - A Simple Hand



A Simple Hand



A Harder One…



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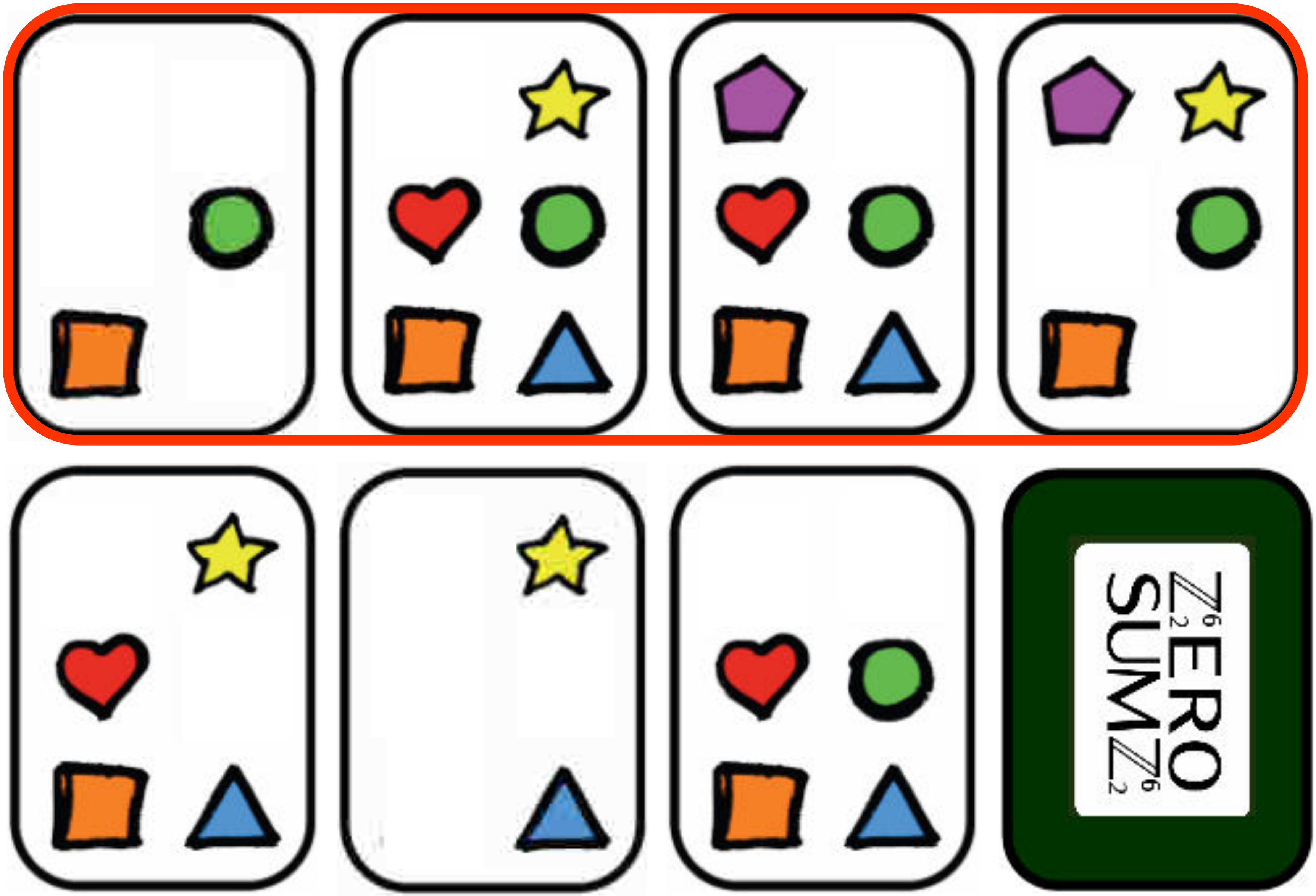


Why must there be always be a Zero Set?

- **Proof #1 (Linear Algebra):**
  - Cards are elements of the vector space Z_2^6
  - 7 vectors from this space => must have a linearly dependent set
  - Some non-trivial linear combination of the 7 vectors sum to the zero vector
  - Scalars in this field are just {0, 1}, so the desired linear combination is just a set of cards

Why must there always be a Zero Set?

- **Proof #2 (For smart kids who have never seen linear algebra):**
  - Denote the card that a collection of cards, S, sums to by Σ(S)
  - There are $2^7 - 1$ different non-empty subsets of the 7 dealt cards
  - Each subset sums to a card in Z_2^6, of which there are 2^6 possibilities.
  - Must be two non-empty subsets, S_1 and S_2, of cards with the same sum, i.e. $\Sigma(S_1) = \Sigma(S_2)$.
  - If S_1 and S_2 are disjoint, their sum is a zero set.
  - If S_1 and S_2 are not disjoint, then they have a non-empty intersection K. Write

$$S_1 = K \cup (S_1 \setminus K),$$
$$S_2 = K \cup (S_2 \setminus K)$$

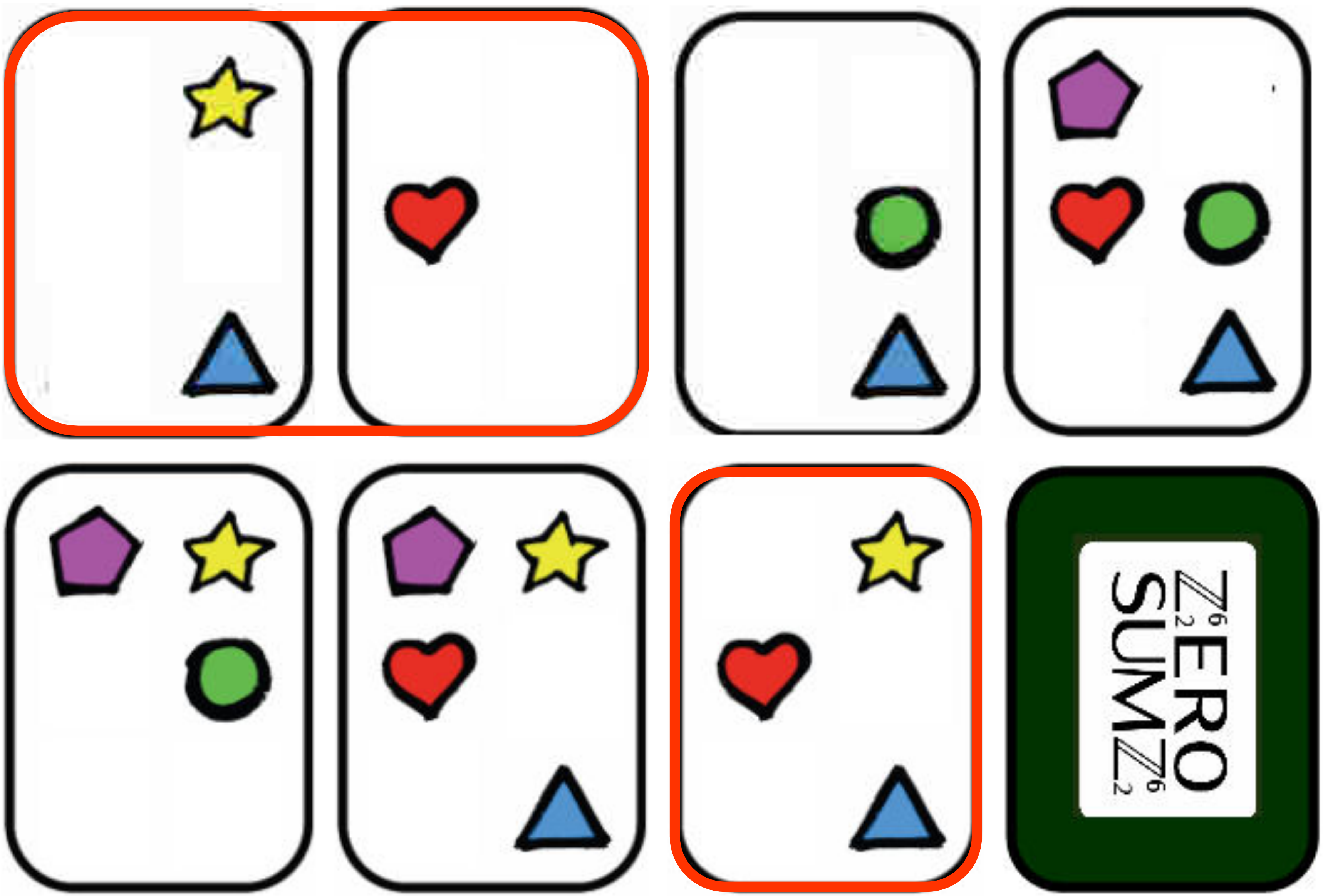
Why must there always be a Zero Set?

- **Proof #2**
  - Case S_1 and S_2 not disjoint: $S_1 = K \cup (S_1 \setminus K)$,

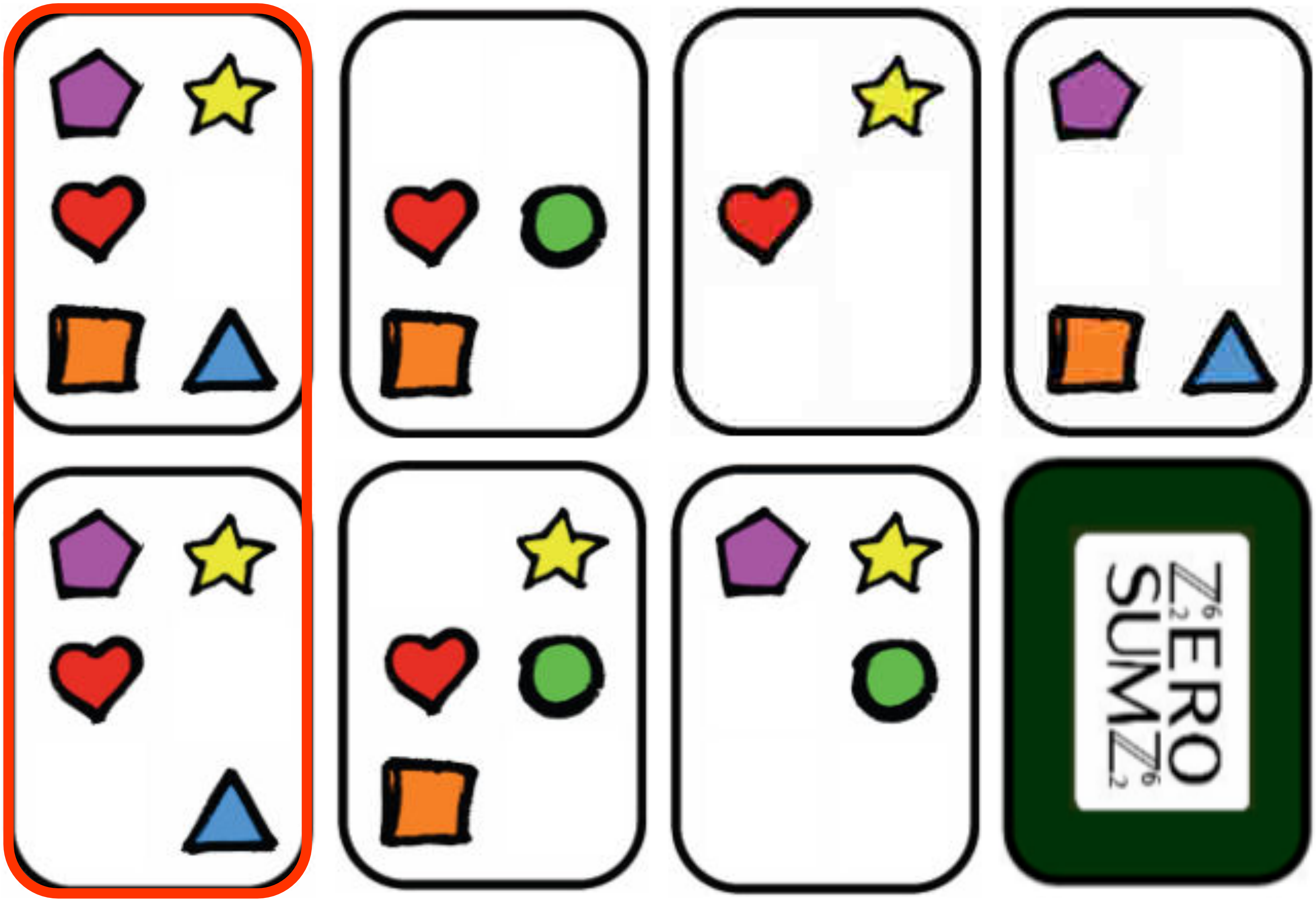
    $S_2 = K \cup (S_2 \setminus K)$
  - Since $\Sigma(S_1) = \Sigma(S_2)$, $\Sigma(S_1) - \Sigma(K) = \Sigma(S_2) - \Sigma(K)$ so that: $\Sigma(S_1 \setminus K) = \Sigma(S_2 \setminus K)$
  - Thus the disjoint union of $S_1 \setminus K$ and $S_2 \setminus K$ forms a zero set.

QED

Easier to understand this proof with actual cards…. (Disjoint case)



Easier to understand this proof with actual cards…. (Non-disjoint case)



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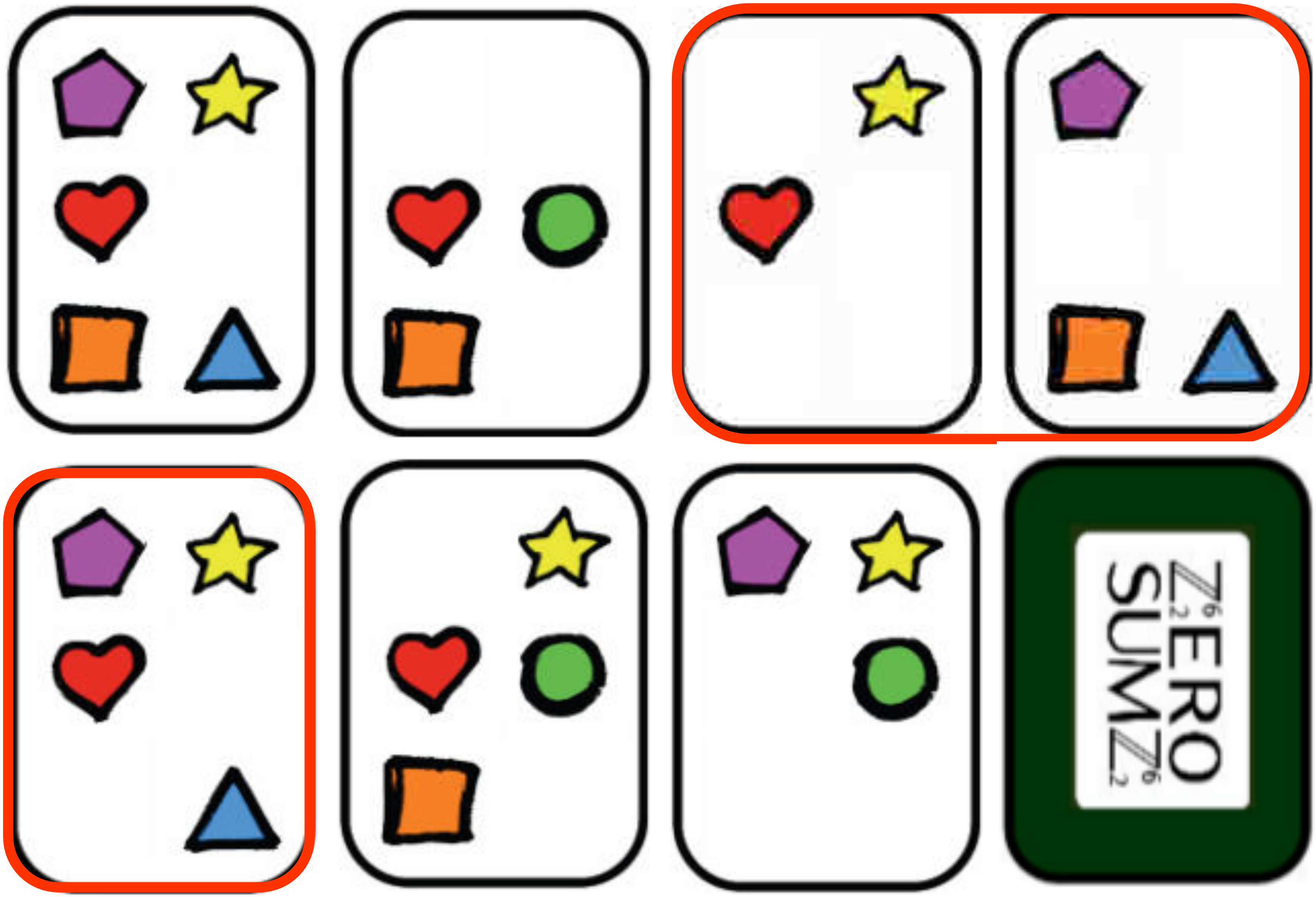

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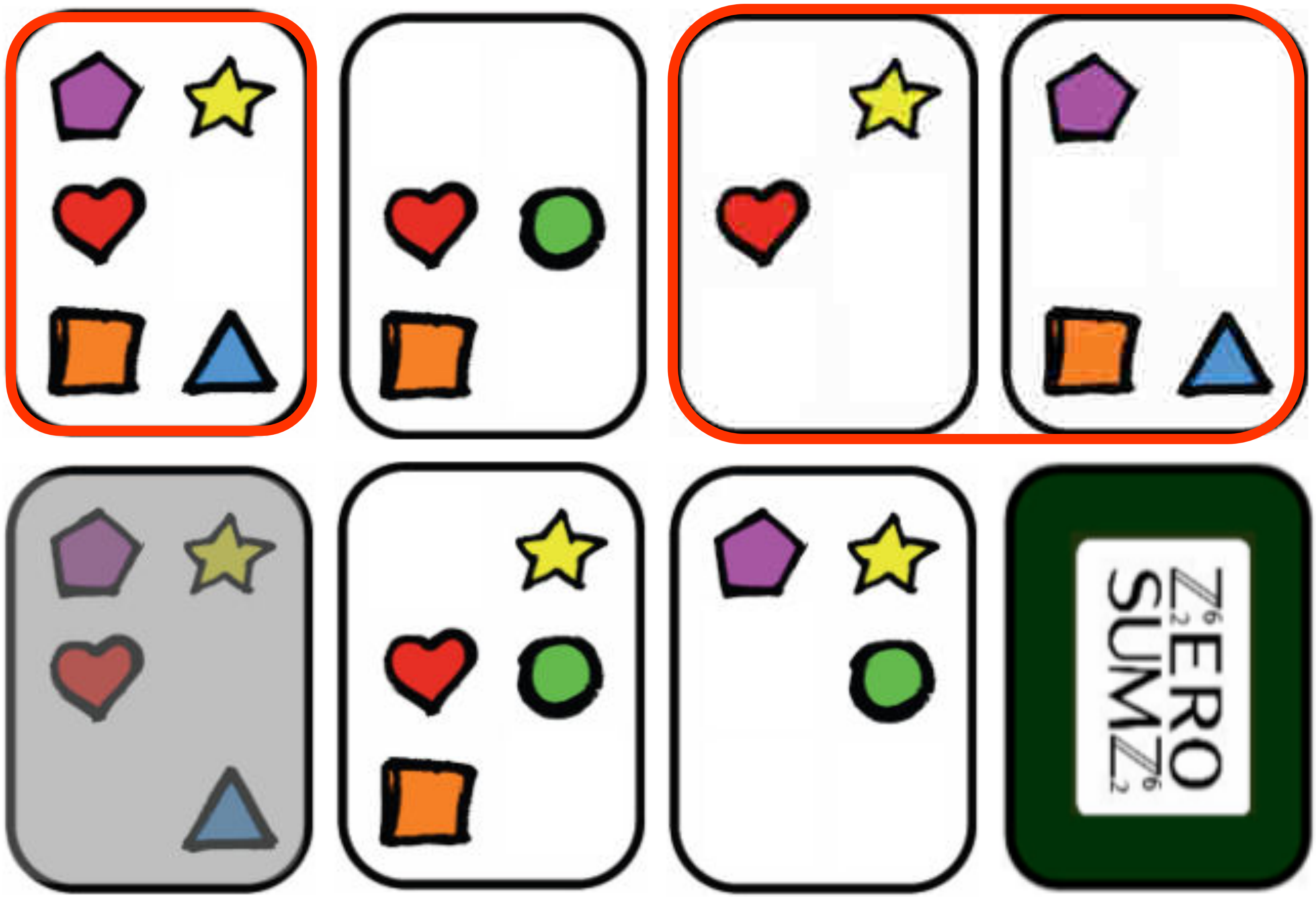
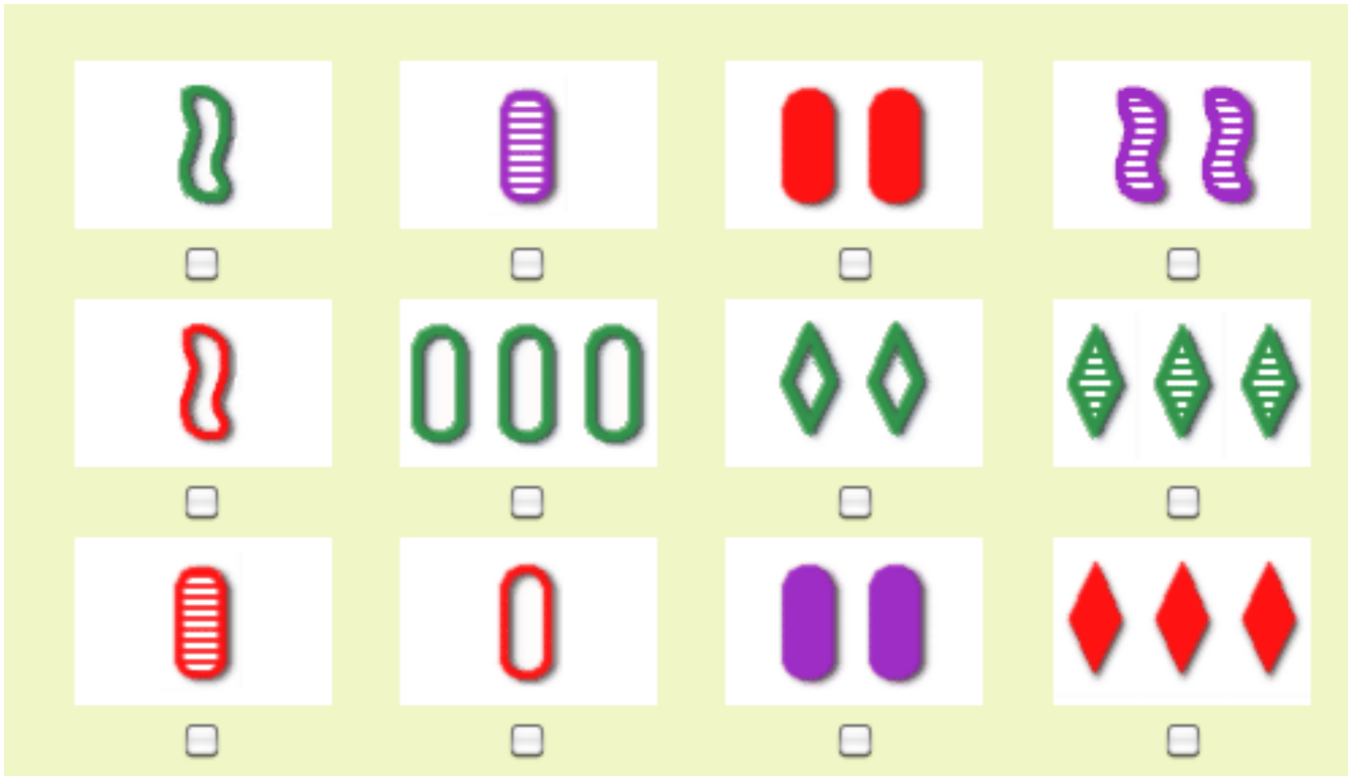


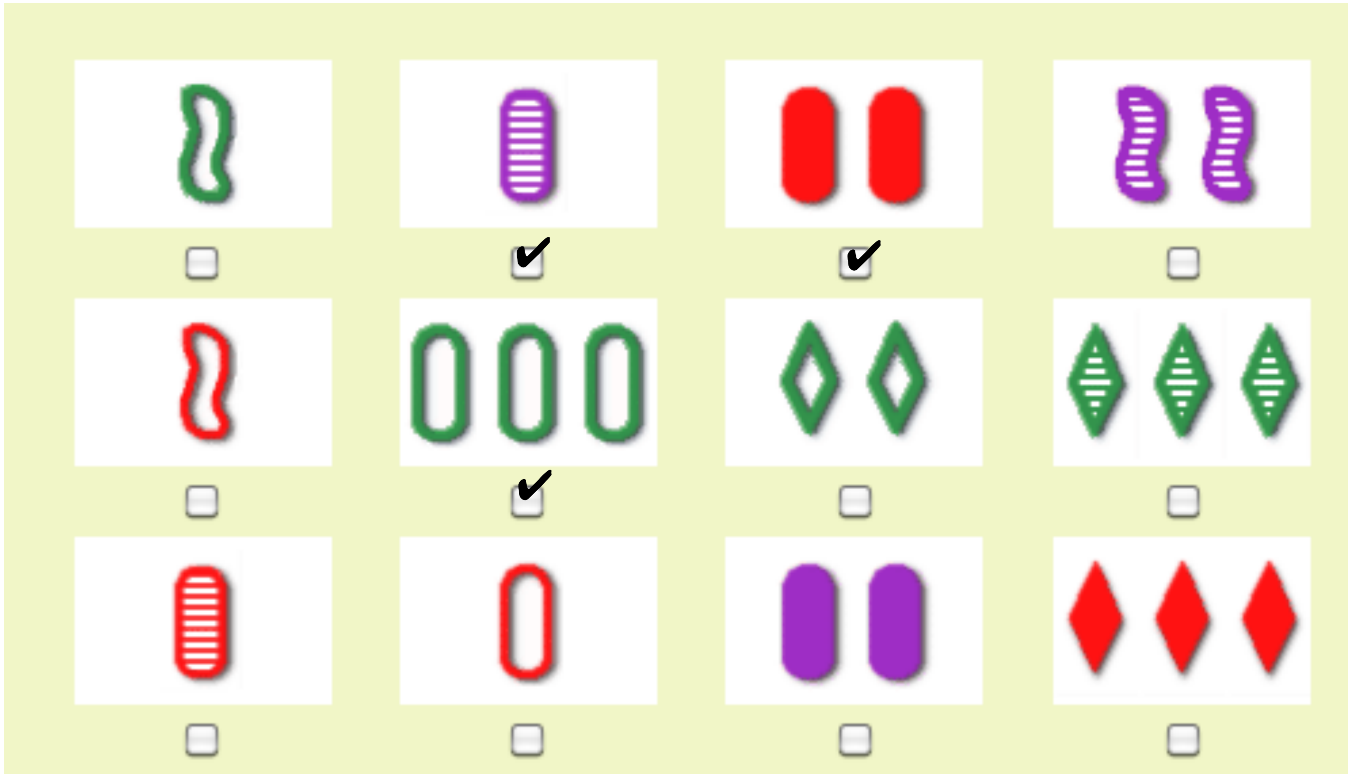
The Card Game Set

- **Invented by population geneticist Marcia Falco while working at Cambridge in 1974**



- **12 cards initially dealt.**
- **Each card has 4 attributes:**
  – color, number of objects, shape and texture
- **A “set” is a collection of cards, such that for each attribute, either all cards are the same or all cards are different**

The Card Game Set



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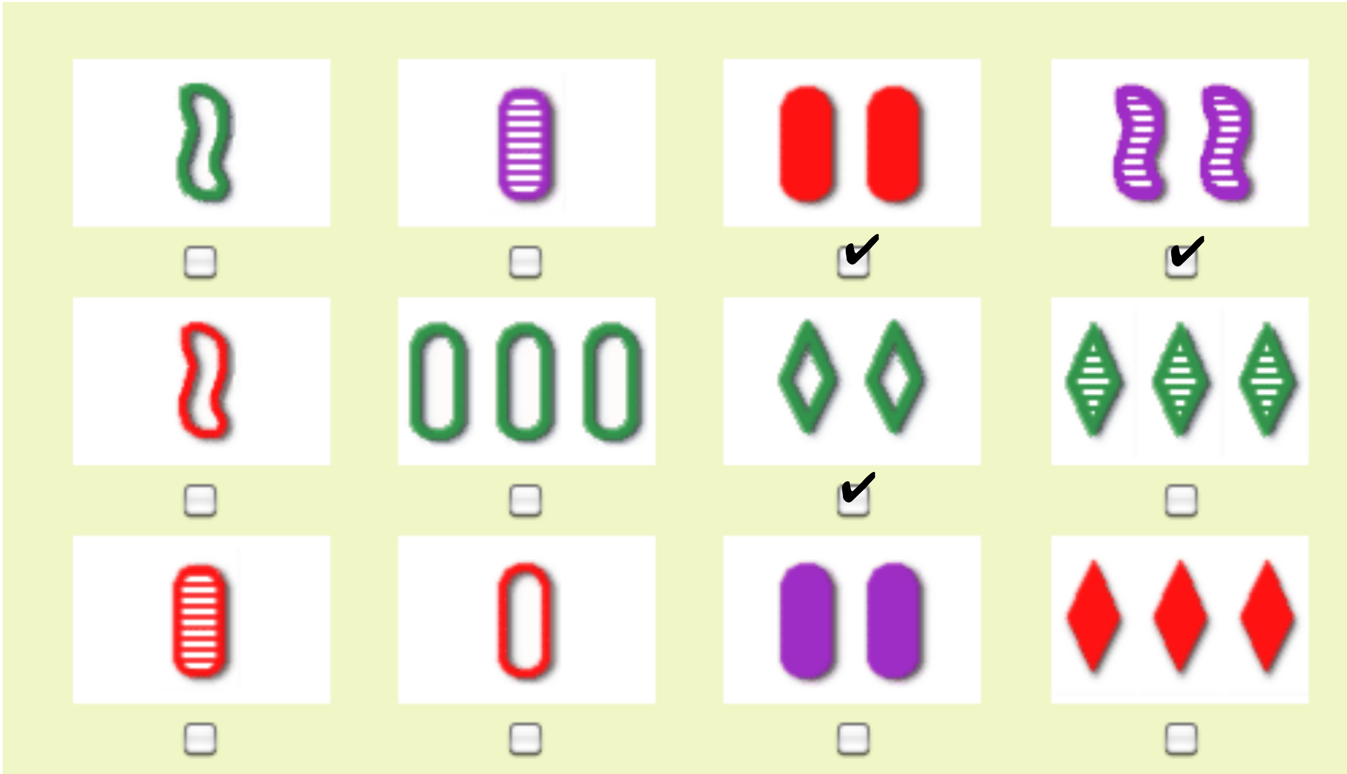

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Why is Set like Zero sumZ?

- **Set is actually a card game over Z_3^4**
- **"Slots" are each of the four attributes {color, number of objects, shape, texture}**
- **Key observation:**
  - The only way for three numbers in {0, 1, 2} to sum to 0 (mod 3) is if all three numbers are the same or all are different!
- **So Set is just Zero sumZ over a different vector space, where the set size must be exactly 3!**

Mathematics of Set

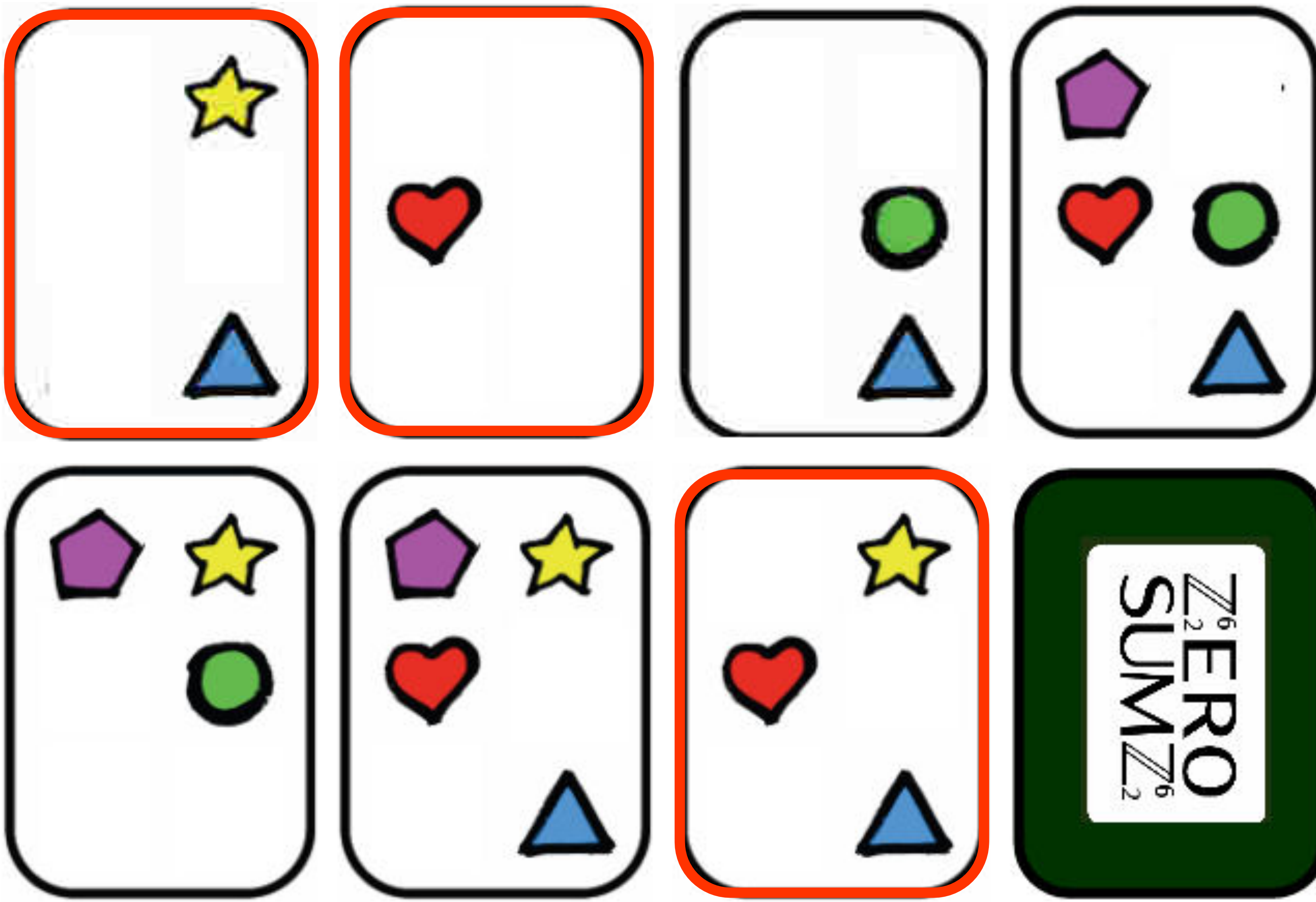
- **In Set, there is not necessarily a "set" in every collection of 12 cards**
- **If all players agree there is no "set" more cards are dealt until some player finds a set**
- **As many as 20 cards can be found with no set**
  – For a construction of such a collection of cards see the Set website
- **Davis and Maclagan showed via a mathematical argument that every collection of 21 cards must contain a set**
- **Maximal collection of cards not containing a set also known, via exhaustive search in the 5D version of Set (i.e. over Z_3^5): 45.**
- **These sets are known as "maximal affine caps." Size of the maximal cap in 6D is known to lie in $112 \leq c_{max} \leq 114$.**
- **In 7D no "respectable" bounds are known**

Mathematics of Set

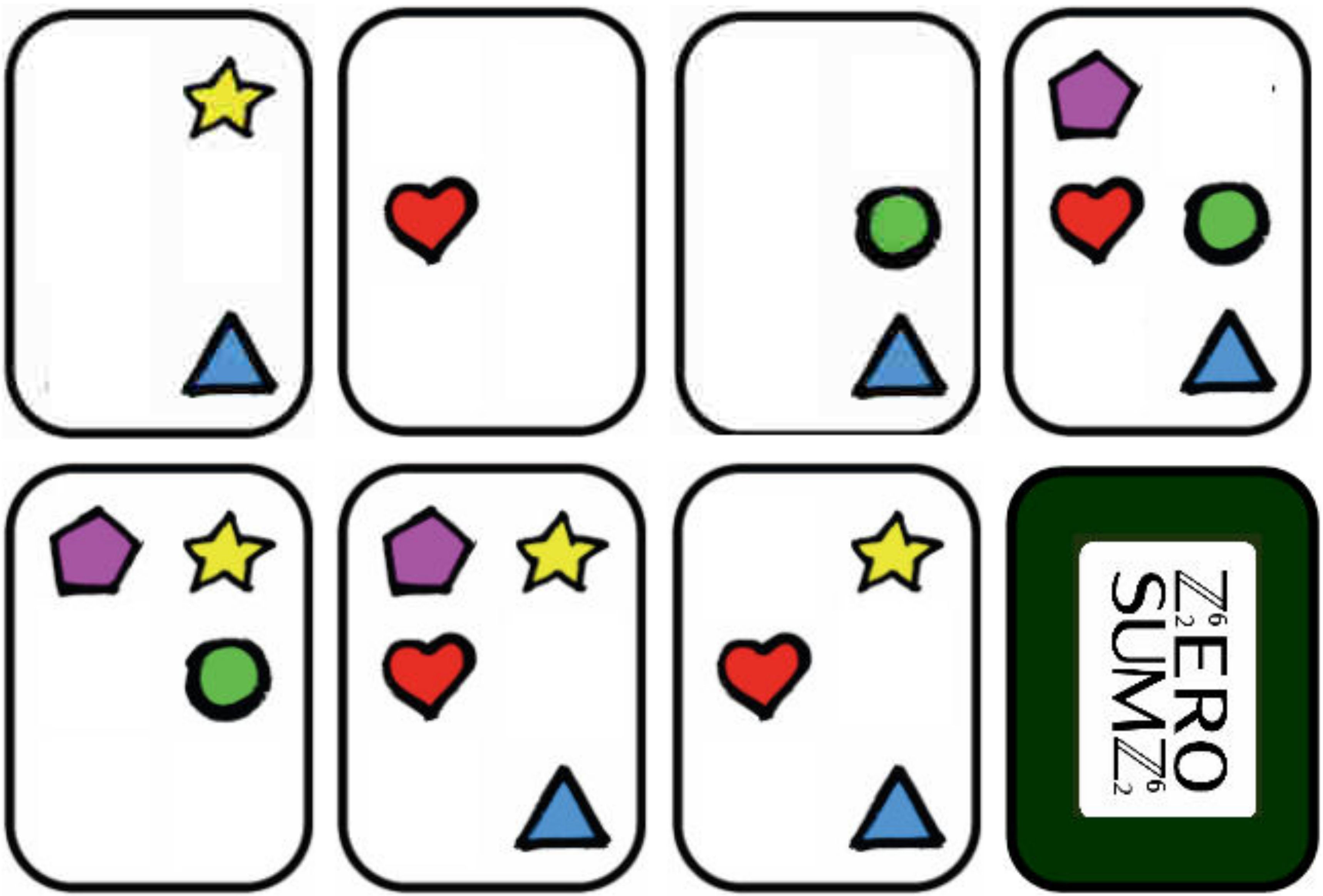
- **Odds against there being no set in a collection of 12 cards has also been studied:**
  – Peter Norvig (http://norvig.com/SET.html) showed that the odds of such an occurrence are approximately 16:1
  – Henrik Warne (http://henrikwarne.com/2011/09/30/set-probabilities-revisited/) showed that the probability of there being no set is not uniform as you play through a deck, ranging from 30:1 in the first round and then quickly leveling off to between 13:1 and 14:1.

Back to Zero sumZ

Returning to our first example hand….



Not the largest possible sum-zero set….



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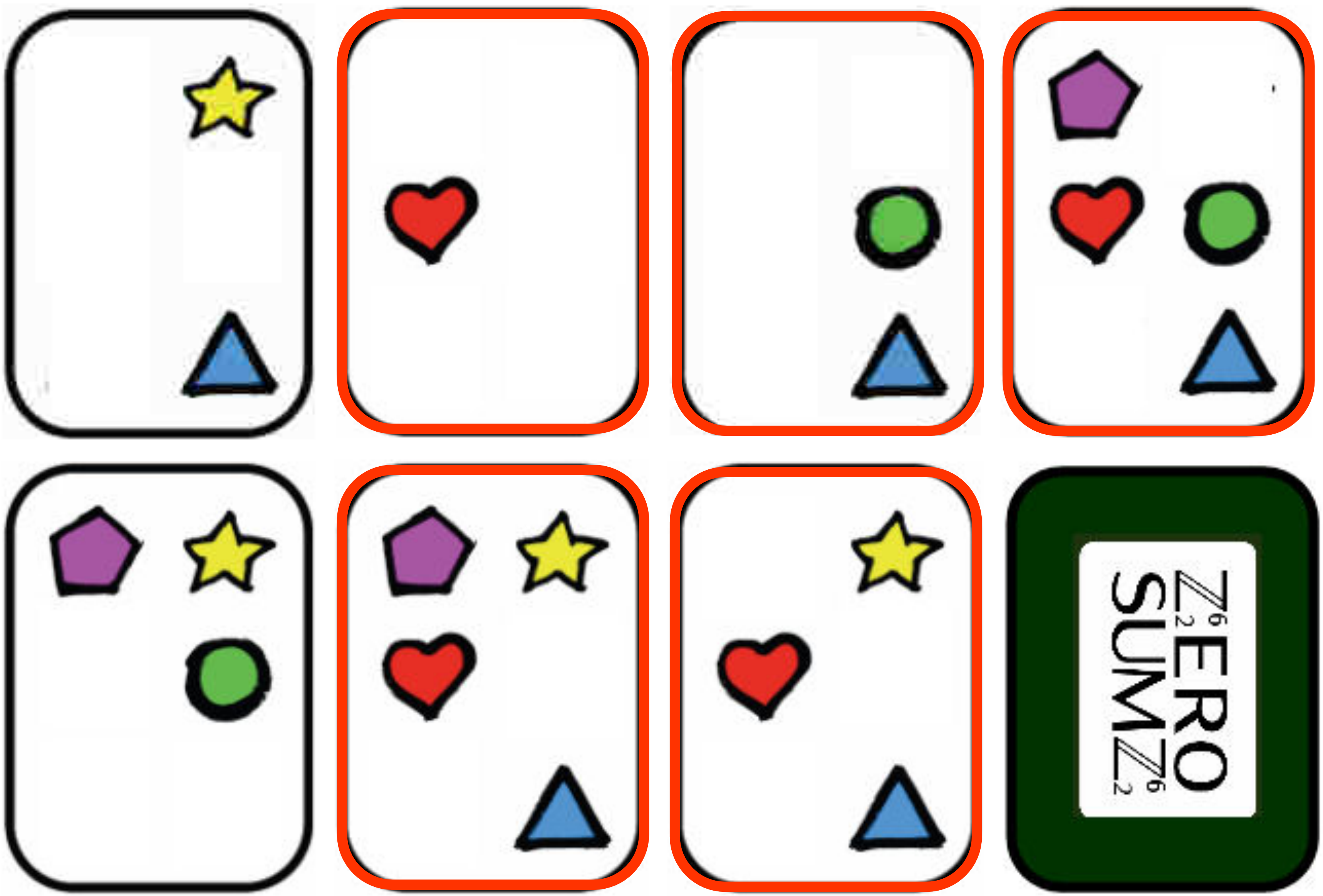


How many zero sum sets can there be?

- Plainly there cannot be exactly 2
- If there are 2 must be at least 3
- With a little work one can show that the number of zero sum sets must be one of {1, 3, 7 or 15} – a consequence of the Nullity + Rank Theorem of Linear Algebra!
- Having more than one zero sum set is fairly common. What is the average number of such sets?
- What is the expected maximum set size?

What do all the cards taken together sum to?

- Corollary: The last deal of 7 or fewer cards must always form a zero set (so not necessary to play this hand – can serve as a checksum)

What is the most likely sum of 3 random cards from the Zero sumZ deck?

What is the least likely sum of 4 random cards from the Zero sumZ deck?

However – does not continue this way forever since the sum of the first 61 cards or first 62 cards is never 0.

When does the alternating property break down?

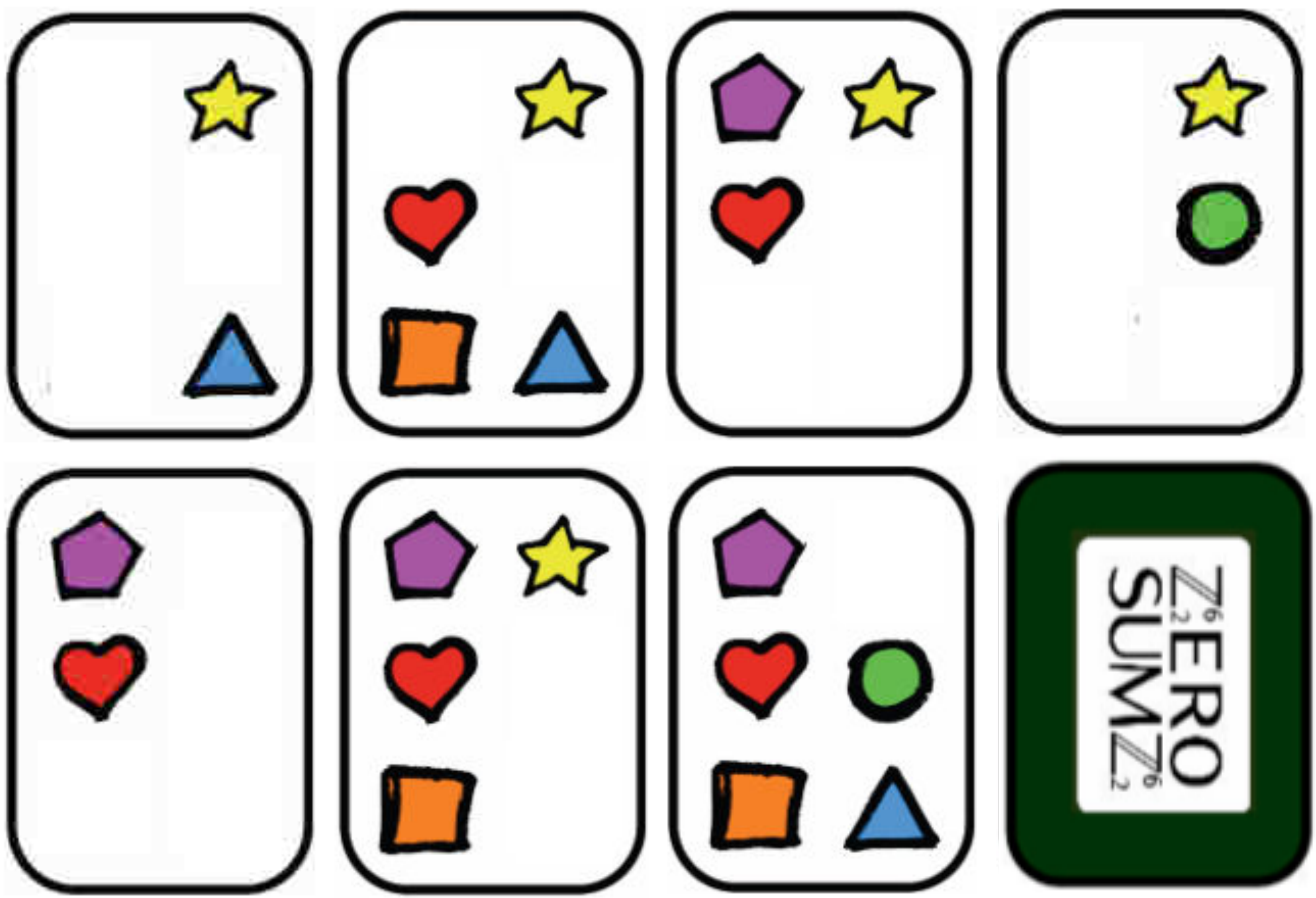
For the $$$

- **Some randomly dealt hands for audience to try….**

A "Grandmaster" Strategy

- **Quickly sum up the value of all cards and work backwards!**

Example:



Other Variations

- **Can imagine playing the same game over Z_3^4. Instead of using the Set attributes, use four disks of different colors that are either empty, 1/3 full, or 2/3 full, and do addition modulo 1.**
- **Because of the different scalar field, there is a difference between addition and subtraction in this case, though a + 2b = a – b so addition and subtraction are the only operations you need to consider.**
- **How many cards at a minimum are required to guarantee a sum-zero set? Of course 5 - by the linear algebra proof! We do not know how to create a variant of the combinatorial proof.**
- **How about over other vector spaces?**
- **Computational complexity of finding linearly dependent sets in finite vector spaces is known to be NP-complete by reduction to 3-SAT – in fact more can be said [see Bhattacharyya et al., 2011]**

Visit http://zerosumz.com to play online or pick up a deck for yourself!